IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Some properties of a uniform fluid sphere in general relativity

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1972 J. Phys. A: Gen. Phys. 5 1305

(http://iopscience.iop.org/0022-3689/5/9/004)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.73
The article was downloaded on 02/06/2010 at 04:39

Please note that terms and conditions apply.

J. Phys. A: Gen. Phys., Vol. 5, September 1972. Printed in Great Britain. © 1972.

Some properties of a uniform fluid sphere in general relativity

A BANERJEE†

Laboratoire de Physique Théorique Associé au CNRS,
Institut Henri Poincaré, Paris 5, France

MS received 24 February 1972, in revised form 5 May 1972

Abstract. We present here a method to obtain a class of exact solutions for uniform fluid spheres surrounded by empty space. It is shown that in some cases there is a reversal in the motion of contraction or expansion of the sphere, while in other cases there is no bouncing at all.

1. Introduction

In the literature there are some well known interior solutions for the adiabatic spherically symmetric motion of a perfect fluid of uniform matter density but nonuniform pressure (Bonnor and Faulks 1967, Bondi 1969, Thompson and Whitrow 1968). In the present paper, however, we consider a specific model of such a sphere surrounded by empty space and discuss its dynamical behaviour. The conditions governing the different states of the sphere's motion can be expressed in terms of the Schwarzschild mass and the initial values of R_0, R_0 and central pressure, where R_0 is the Schwarzschild radius of the sphere. In some cases the motion of contraction or expansion is reversed and in certain other cases the motion of the sphere has no bounce at all. There are, however, no oscillations in the model under consideration. Matching of the interior solutions with the exterior Schwarzschild metric at the boundary can be done by the procedure of Raychaudhuri (1953).

2. Integration of the field equations and the conditions for a realistic model

We consider the line element in the isotropic form

$$ds^2 = e^{\nu}\, dt^2 - e^{\mu}(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2) \tag{1}$$

where ν and μ are functions of time as well as the radial coordinate. The components of the energy–momentum tensor are

$$T_1^1 = T_2^2 = T_3^3 = -p \qquad T_4^4 = \rho \qquad T_{14} = 0 \tag{2}$$

because the pressure is isotropic and comoving coordinates are used. The divergence

† Postal address: 11, rue Pierre Curie, Paris 5, France. Permanent address: Department of Physics, Jadavpur University, Calcutta 32, India.

relations $T^{\mu\nu}_{;\nu} = 0$ gives two equations

$$p' = -\tfrac{1}{2}(\rho+p)\nu' \tag{3}$$

and

$$\dot{\rho} = -\tfrac{3}{2}(\rho+p)\dot{\mu}. \tag{4}$$

Integrating the field equations one obtains, in view of the fact that $\rho = \rho(t)$

$$e^{-\nu}\dot{\mu}^2 = A(t) \tag{5}$$

where $A(t)$ is an arbitrary function of time. Also

$$e^{\mu} = \frac{4f}{(fr^2+\Phi/4)^2} \tag{6}$$

where f and Φ are arbitrary functions of time alone and

$$\Phi + A = \tfrac{32}{3}\pi\rho. \tag{7}$$

One of the functions Φ and A is, however, determined if the matter distribution is known. It is evident from (3), (5) and (6) in view of Raychaudhuri's argument (1955) that at any instant ν' and consequently p' are of the same sign everywhere. We are, however, interested in the case where pressure monotonically decreases with r inside the fluid sphere and vanishes at its boundary ($r = r_0$).

Since $p = 0$ at $r = r_0$, one can get after integration of (4) at the boundary

$$\rho = \beta\, e^{-\frac{3}{2}\mu_0} \tag{8}$$

that is

$$\rho = \tfrac{1}{8}\beta\frac{(fr_0^2+\Phi/4)^3}{f^{3/2}} \tag{9}$$

where $\beta = (3/4\pi)(m/r_0^3)$, m being the usual Schwarzschild mass of the sphere. It follows, therefore, from (3), (4), (5), (6) and (7) that:

$$e^{-\nu} = \left(\frac{4\pi}{3}\beta\frac{(fr_0^2+\Phi/4)^3}{f^{3/2}} - \Phi\right)(fr^2+\Phi/4)^2\left(\frac{\dot{f}}{f}\frac{\Phi}{4} - \frac{\dot{\Phi}}{2} - \dot{f}r^2\right)^{-2} \tag{10}$$

$$\rho + p = \frac{1}{16\pi}\frac{(\dot{\Phi}+\dot{A})(fr^2+\Phi/4)}{\frac{1}{2}\dot{\Phi} - \frac{1}{4}(\dot{f}/f)\Phi + \dot{f}r^2} \tag{11}$$

and

$$p' = \frac{\dot{\Phi}+\dot{A}}{4\pi}\frac{(f\dot{\Phi}-\Phi\dot{f})r}{\{\dot{\Phi} - \frac{1}{2}(\dot{f}/f)\Phi + 2\dot{f}r^2\}^2}. \tag{12}$$

Since p' is negative

$$(\dot{\Phi}+\dot{A})(f\dot{\Phi}-\Phi\dot{f}) < 0. \tag{13}$$

In the following we have considered only positive values of Φ; f is necessarily positive in view of (6). Remembering the relation (13) and the fact that $(\rho+p)$ is always positive even at the origin ($r = 0$) we get

$$2\frac{\dot{\Phi}}{\Phi} > \frac{\dot{f}}{f} > \frac{\dot{\Phi}}{\Phi} \tag{14}$$

for contraction where $\dot{\rho} > 0$, or in other words $\dot{\mu} < 0$. In this case both $\dot{\Phi}/\Phi$ and $\dot{f}/f$ are positive. Again for expansion ($\dot{\rho} < 0$ or $\dot{\mu} > 0$) we get by the same argument

$$2\frac{\dot{\Phi}}{\Phi} < \frac{\dot{f}}{f} < \frac{\dot{\Phi}}{\Phi} \tag{14'}$$

where both $\dot{\Phi}/\Phi$ and $\dot{f}/f$ are negative. The relations (14) and (14′) can also be written in the form

$$2\left|\frac{\dot{\Phi}}{\Phi}\right| > \left|\frac{\dot{f}}{f}\right| > \left|\frac{\dot{\Phi}}{\Phi}\right|. \tag{15}$$

3. Behaviour in a special case

Let us consider a relation

$$f = \Phi^{\alpha}$$

that is (16)

$$\frac{\dot{f}}{f} = \alpha\frac{\dot{\Phi}}{\Phi}$$

where α is a constant and $1 < \alpha < 2$ from (15). Thus using (16) one gets

$$e^{\mu} = \frac{4\Phi^{\alpha}}{(\Phi^{\alpha}r^2 + \Phi/4)^2} \tag{17}$$

$$\rho = \tfrac{1}{8}\beta\frac{(\Phi^{\alpha}r_0^2 + \Phi/4)^3}{\Phi^{3\alpha/2}}. \tag{18}$$

Again for convenience normalizing the value of e^{ν} at the origin ($r = 0$) so that $e^{\nu} = 1$ at $r = 0$ (that is, identifying t as the proper time of an observer permanently at the centre) we get

$$e^{\nu} = \left(\frac{\alpha}{2-\alpha}r^2 + \frac{1}{4\Phi^{(\alpha-1)}}\right)^2\left(r^2 + \frac{1}{4\Phi^{(\alpha-1)}}\right)^{-2} \tag{19}$$

where Φ satisfies the relation

$$\frac{4\pi}{3}\beta\Phi^{3\alpha/2}\left(r_0^2 + \frac{1}{4\Phi^{(\alpha-1)}}\right)^3 - \Phi = \frac{(2-\alpha)^2\dot{\Phi}^2}{\Phi^2} \tag{20}$$

which implies

$$\frac{4\pi}{3}\beta\Phi^{(\frac{3}{2}\alpha-1)}\left(r_0^2 + \frac{1}{4\Phi^{(\alpha-1)}}\right)^3 \geqslant 1. \tag{21}$$

The equality sign corresponds to $\dot{\Phi}/\Phi^{3/2} = 0$ which, in other words, means $\dot{\Phi} = 0$ for finite values of Φ. Calling

$$\frac{4\pi}{3}\beta\Phi^{(\frac{3}{2}\alpha-1)}\left(r_0^2 + \frac{1}{4\Phi^{(\alpha-1)}}\right)^3 = y \tag{22}$$

we easily get

$$\frac{\mathrm{d}y}{\mathrm{d}\Phi} = \frac{4\pi}{3}\beta\left(r_0^2 + \frac{1}{4\Phi^{(\alpha-1)}}\right)^2 \Phi^{(\frac{3}{2}\alpha-2)}\left(\frac{3\alpha}{2}-1\right)\left(r_0^2 - \frac{1}{4\Phi^{(\alpha-1)}}\frac{(\frac{3}{2}\alpha-2)}{(\frac{3}{2}\alpha-1)}\right). \tag{23}$$

It is evident from (22) that for $\alpha > \frac{4}{3}$, 'y' tends to infinity as Φ approaches zero or infinity and (23) shows that $\mathrm{d}y/\mathrm{d}\Phi = 0$ only when

$$r_0^2 = \frac{1}{4\Phi^{(\alpha-1)}}\frac{(\frac{3}{2}\alpha-2)}{(\frac{3}{2}\alpha-1)}.$$

The minimum value of 'y' is thus given by

$$y_{\min} = \frac{4\pi}{3}\beta\left(\frac{1}{4r_0^2}\frac{(\frac{3}{2}\alpha-2)}{(\frac{3}{2}\alpha-1)}\right)^{(\frac{3}{2}\alpha-1)/(\alpha-1)}\left(\frac{3(\alpha-1)r_0^2}{\frac{3}{2}\alpha-2}\right)^3. \tag{24}$$

So for $\alpha > \frac{4}{3}$ three different situations arise according to whether $y_{\min}$ is greater than, equal to, or less than 1. These different conditions are, however, dependent on different relative values of the Schwarzschild mass (m) and the comoving radius of the boundary (r_0).

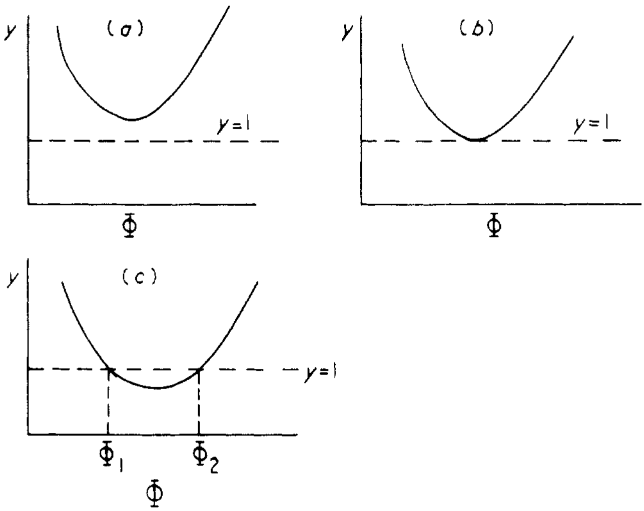

Figure 1. Curve of y against Φ for three different cases. (a) $y_{\min} > 1$; (b) $y_{\min} = 1$; (c) $y_{\min} < 1$.

3.1. Case (i) (figure 1a)

$y_{\min} > 1$ and in view of (20) $\dot{\Phi}$ is real and nonzero for all finite values of Φ. The y–Φ curve does not touch or intersect the line $y = 1$ at all. So in such a case the system may start from the infinite dilution stage ($\Phi \to 0$) and collapse to the singularity of zero proper volume (infinite proper density, $\Phi \to \infty$) or the reverse may also occur depending on the sign of the initial $\dot{\Phi}$.

3.2. Case (ii) (figure 1b)

$y_{\min} = 1$ and in view of (20) $\dot{\Phi} = 0$ only at the point where the y–Φ curve touches the line $y = 1$, otherwise it is real and nonzero. The system at this stage ($\dot{\Phi} = 0$) is in unstable equilibrium. The slightest perturbation in the direction of contraction will lead to a catastrophic collapse or the slightest expansion will finally lead to the infinite dilution stage.

3.3. *Case (iii) (figure 1c)*

$y_{min} < 1$ and $\dot{\Phi}$ is imaginary between Φ_1 and Φ_2 where Φ_1 and Φ_2 are the values of Φ at the points of intersection of the y–Φ curve with the line $y = 1$. These points of intersection are the turning points of the motion of the sphere because here $\dot{\Phi}$ changes sign. So either the system may start from infinite dilution, contract and bounce back; or the system explodes from the initial singularity of infinite proper density, reaches the maximum volume and again collapses to the singularity.

Again for $\alpha < \frac{4}{3}$, 'y' increases with Φ and the curve intersects the line $y = 1$ only at a single point where $\dot{\Phi} = 0$. Thus the system may explode from the initial singularity and again collapse to zero proper volume after reaching the turning point. But on the other hand the system which once starts to contract cannot be halted and the collapse continues up to the final singularity. The case $\alpha < \frac{4}{3}$, however, is to be distinguished from the case $\alpha = \frac{4}{3}$. In the former case $\dot{\Phi} = 0$ (that is $\dot{R}_0 = 0$) for some finite time, while in the latter case it is not necessarily so depending on the initial conditions, because in this case as $\Phi \to 0$, 'y' tends to some finite quantity $\pi/48\beta$.

4. Matching of the interior solutions to the Schwarzschild metric

The interior solutions given above can be shown to be continuous at the boundary with the outside Schwarzschild metric by the procedure of Raychauduri (1953). Details of such a procedure are omitted here.

Putting $e^{\mu}r^2 = \xi^4$ and $x = \ln r$ one can obtain on integration of the field equations remembering that the metric and its first derivatives are continuous at the boundary and also that there is no singularity at the origin ($r = 0$)

$$\left(\frac{\partial \xi}{\partial x}\right)^2 = \tfrac{1}{4}\xi^2 + \tfrac{1}{16}A(t)\xi^6 - \tfrac{1}{2}m \tag{25}$$

m being the Schwarzschild mass calculated in the form $m = \frac{4}{3}\pi\rho\xi_0^6$ and $A(t)$ being equal to $e^{-\nu}\dot{\mu}^2$.

Therefore the field in empty space ($r \geqslant r_0$) is given by

$$e^{\mu} = \frac{\xi^4}{r^2} \qquad e^{\nu} = \left(\frac{4\dot{\xi}}{A^{1/2}\xi}\right)^2$$

where ξ is the solution of equation (25). The transformation to the Schwarzschild metric can be obtained by the relations

$$R = \xi^2 = e^{\frac{1}{2}\mu}r$$

and

$$dT = \left(\frac{8r\dot{\xi}\xi'}{A(t)(\xi^2 - 2m)}\right) dt + \frac{\xi^6 A(t)}{2r(\xi^2 - 2m)} dr$$

where R and T are the well known Schwarzschild radial and time coordinates.

5. Conclusions

The present paper considers what is, in effect, a special case of a more general equation derived by Thompson and Whitrow (1968 to be referred to as TW). They may be compared if one identifies the symbols μ, ν, Φ, f, $1/r_0$ and β of our paper with λ, ν, $16BC$, $4C^2$, k and $3k^6K^2/8\pi$ respectively of TW. The special case which we consider in equation (16) of our paper is equivalent to a relation between B and C of TW in the form

$$C = lB^{1+1/n} \tag{26}$$

where $l = 2^{2+3/n}$ and 'n' can be expressed in the symbols used in our paper as $n = (2-\alpha)/2(\alpha-1)$. In TW an equation in the physical parameters of the system is derived and the specific solutions are obtained from the consideration that the material at the centre obeys the polytropic equation pv^γ = constant, where v is an element of volume and γ is a constant. In view of the relation (26) the Schwarzschild radius of the sphere in our case R_0 is given in the notation of TW as

$$R_0 = \Theta\frac{(u/n-1)^{1+n}}{u/n} \tag{27}$$

where $u = \rho/p$ is evaluated at the centre and Θ is a constant. Θ can in turn be identified with $(1/2r_0^3)(4r_0^2)^{(3\alpha-2)/2(\alpha-1)}$ in our paper, where r_0 is the comoving radius of the boundary of the sphere. In view of (27) the polytropic index γ in our case turns out to be equal to $\{1+(1/3n)+(2/3u)\}$ which is no longer a constant. Substituting (27) in the equation (37) of TW and integrating yields

$$n^2\Theta^2\dot{V}^2 = V^{1-2n}\left(\frac{2m}{\Theta}\frac{(V+1)^3}{V^{n+2}}+\frac{V}{h^2}-\frac{2(1+V)}{h}\right) \tag{28}$$

where $V = (u/n-1)$. However, since the equation (16) of our paper is equivalent to a condition on the integral of (37) of TW, it follows that the integration constant h must be equal to $\frac{1}{2}$ and the condition that h is not arbitrary in our case implies that the initial data (that is R_0, $\dot{R}_0$, central pressure) cannot be specified arbitrarily resulting in the fact that oscillations are ruled out.

On putting $h = \frac{1}{2}$ in (28) we get

$$\Theta^3n^2\dot{V}^2 = 2mV^{1-2n}\left(\frac{(V+1)^3}{V^{n+2}}-\frac{2\Theta}{m}\right) \tag{29}$$

which is exactly equivalent to (20) of our paper.

It is to be noted that one can distinguish the various cases of motion (cases (i), (ii) and (iii)) discussed previously, in terms of the Schwarzschild mass and the initial values of R_0, $\dot{R}_0$ and central pressure as well. For instance $y_{\min}$ in equation (24) of our paper is equivalent to $(27m/2\Theta)\,(1-n)^{n-1}/(n+2)^{n+2}$ where Θ can be obtained from the relation (27) in terms of the Schwarzschild mass and the initial values of R_0 and the central pressure and the three different cases mentioned previously correspond to the condition that $(27m/2\Theta)(1-n)^{n-1}/(n+2)^{n+2}$ is greater than, equal to, or less than 1.

Acknkowledgments

The author wishes to thank CNRS, France, for financial aid during the work. He is grateful to Professor A Papapetrou and Dr J Madore for their hospitality. He is also grateful to the referees for their very valuable comments on this paper.

References

Bonnor W B and Faulks M C 1967 *Mon. Not. R. Astronom. Soc.* **137** 239–51
Bondi H 1969 *Mon. Not. R. Astronom. Soc.* **142** 333–53
Raychaudhuri A K 1953 *Phys. Rev.* **89** 417
—— 1955 *Z. Astrophys.* **37** 103–7
Thompson I H and Whitrow G J 1968 *Mon. Not. R. Astronom. Soc.* **139** 499